

On the invariant symmetries of the \mathcal{D} -metrics

Joan Josep Ferrando¹ and Juan Antonio Sáez²

Abstract

We analyze the symmetries and other invariant qualities of the \mathcal{D} -metrics (type D aligned Einstein Maxwell solutions with cosmological constant whose Debever null principal directions determine shear-free geodesic null congruences). We recover some properties and deduce new ones about their isometry group and about their quadratic first integrals of the geodesic equation, and we analyze when these invariant symmetries characterize the family of metrics. We show that the subfamily of the Kerr-NUT solutions are those admitting a Papapetrou field aligned with the Weyl tensor.

PACS: 04.20.Cv, 04.20.-q

¹ Departament d'Astronomia i Astrofísica, Universitat de València, E-46100 Burjassot, València, Spain. E-mail: joan.ferrando@uv.es

² Departament de Matemàtiques per a l'Economia i l'Empresa, Universitat de València, E-46022 València, Spain. E-mail: juan.a.saez@uv.es

1 Introduction

Explicit integration of the Einstein-Maxwell equations with cosmological constant has been achieved by several authors for the family of aligned type D metrics whose null principal directions define shear-free geodesic congruences^{1,2,3} (see also references therein). This family of solutions has been named the \mathcal{D} -metrics.³ They can be deduced from the Plebański and Demiański¹ line element by means of several limiting procedures (see the Stephani *et al.* book⁴ and references therein, and the recent paper by Griffiths and Podolský⁵ for a detailed analysis).

The family of the \mathcal{D} -metrics contains significant and well-known solutions of the Einstein equations, like the Reissner-Nordström and the Kerr-Newman black holes and their vacuum limit, the Schwarzschild and Kerr solutions, as well as, other well-known space-times that generalize them, such as the charged Kerr-NUT solutions. The generalized C-metrics, that describe two accelerated black holes moving in opposite directions,⁶ also belong to this family.

Elsewhere⁷ we have given the Rainich theory for type D aligned Einstein-Maxwell solutions with (or without) cosmological constant. The interesting nature of this study is that offers an intrinsic and explicit characterization of the \mathcal{D} -metrics by means of algebraic conditions on the curvature tensor. Now, in the present work we analyze another important quality of the \mathcal{D} -metrics: they admit several *invariant symmetries*.

Although the existence of these symmetries can be shown starting from the plain expression of the metric line element, their invariant character and their close relation with the curvature tensor become more evident if one presents them without explicit integration of the Einstein-Maxwell equations. We can quote some historical references that afford results in this sense.

The existence of a 2+2 conformal Killing tensor in type D vacuum metrics and in their charged counterpart was shown by Walker and Penrose.⁸ They also proved that a full Killing tensor exists in the charged Kerr black hole. Starting from this result Houghston and Sommers⁹ studied the general conditions under which this Killing tensor may be constructed.

In a subsequent work, Houghston and Sommers¹⁰ proved that the symmetries of these space-times arise from properties of the curvature tensor. They gave the expression of a complex Killing vector as a (differential) concomitant of an aligned conformal Killing-Yano bivector, and they showed that this Killing vector degenerates (it defines a unique real Killing vector) if, and only if, the metric admits a Killing tensor.¹⁰ Moreover, in this degenerate case, at least one other Killing vector exists that commutes with the first one.

At this point a paper by Collinson and Smith¹¹ showed that, under the assumptions of the papers by Houghston and Sommers, if a Killing tensor exists, then the space-time also admits a Killing-Yano 2-form. A similar result was also obtained by Stephani.¹²

In this work we revisit all these invariant symmetries of the \mathcal{D} -metrics and we analyze how these geometric properties characterize this family of space-times. We also point out that the inheriting property of the (source) electromagnetic field is related to the commutativity of the 2-dimensional isometry group that the \mathcal{D} -metrics admit in accord with a result by Michalski and Wainwright.¹³ Moreover, we show that this feature can be stated as a condition on the curvature tensor.

Another interesting result of this work concerns the Papapetrou fields defined by the Killing vectors. If ξ is a Killing vector, the *Killing 2-form* $\nabla\xi$ is closed and, in the vacuum case, it is a solution of the source-free Maxwell equations. Because this fact was pointed

out by Papapetrou,¹⁴ the covariant derivative $\nabla\xi$ has also been called the *Papapetrou field*.¹⁵

Metrics admitting an isometry were studied by considering the properties of the associated Killing 2-form,^{16,17} and this approach was extended to the spacetimes with an homothetic motion.^{18,19} The vacuum solutions with an isometry have been classified by considering the algebraic structure of the Killing 2-form,^{15,20} and several extensions have been developed for homothetic and conformal motions^{21,22} and for non vacuum solutions.²³

In the Kerr geometry the principal directions of the Killing 2-form associated with the timelike Killing vector coincide with the two double principal null (Debever) directions of the Weyl tensor.¹⁵ This means that the Killing 2-form is a Weyl principal bivector. This fact has been remarked upon by Mars²⁴ who has also shown that it characterizes the Kerr solution under asymptotic flatness behavior.

A question naturally arises: can all the vacuum solutions with this property of the Kerr metric be determined? Elsewhere^{25,26} we have shown that the Petrov-Bel type I vacuum space-times with this characteristic admit a 3-dimensional group of isometries of Bianchi type I or II and that a close relationship exists between the Weyl principal directions and the isometry group.

In the present work we fulfil a similar study for Petrov-Bel type D metrics. Thus, we show here that the Kerr-NUT solutions are the type D vacuum metrics with a non null Killing 2-form aligned with the Weyl geometry. Actually this property also holds for the charged Kerr-NUT solutions with cosmological constant.

The paper is organized as follows. In Sec. II we introduce the notation and some useful concepts about 2+2 space-time structures and non null Maxwell fields that we need in the work. The non conformally flat metrics admitting a conformal Killing-Yano 2-form are analyzed and characterized in Sec. III. In Sec. IV we show that if the metric is invariant under the divergence of this conformal Killing-Yano tensor, then both the Ricci and Weyl tensor have the same algebraic type as the \mathcal{D} -metrics. In Sec. V we study several equivalent conditions, related to the invariance of the electromagnetic field, that satisfy the \mathcal{D} -metrics and we show that these properties also hold for a slightly wider family of space-times. In Sec. VI we offer two characterizations of the \mathcal{D} -metrics, one of them based on the properties of the conformal Killing-Yano tensor. The other one is intrinsic and explicit and imposes conditions on the curvature tensor. Finally, Sec. VII is devoted to analyzing the Kerr-NUT space-times in detail and to studying the equivalence of several geometric properties that characterize them, like that of admitting an aligned Papapetrou field.

2 Some notation and useful concepts

Let (V_4, g) be an oriented space-time of signature $\{-, +, +, +\}$. The metric G on the space of 2-forms is $G = \frac{1}{2}g \wedge g$, \wedge denoting the double-forms exterior product, $(A \wedge B)_{\alpha\beta\mu\nu} = A_{\alpha\mu}B_{\beta\nu} + A_{\beta\nu}B_{\alpha\mu} - A_{\alpha\nu}B_{\beta\mu} - A_{\beta\mu}B_{\alpha\nu}$. If F and H are 2-forms, (F, H) denotes the product with G :

$$(F, H) \equiv G(F, H) = \frac{1}{4}G_{\alpha\beta\lambda\mu}F^{\alpha\beta}H^{\lambda\mu} = \frac{1}{2}F_{\alpha\beta}H^{\alpha\beta}.$$

If A and B are two 2-tensors, we denote $A \cdot B$ the tensor with components $(A \cdot B)_{\alpha\beta} = A_{\alpha}^{\mu}B_{\mu\beta}$. Moreover, $[A, B]$ and $\{A, B\}$ denote the commutator and anti-commutator,

respectively:

$$[A, B] = A \cdot B - B \cdot A, \quad \{A, B\} = A \cdot B + B \cdot A.$$

A self-dual 2-form is a complex 2-form \mathcal{F} such that $*\mathcal{F} = i\mathcal{F}$, where $*$ is the Hodge dual operator. We can associate biunivocally with every real 2-form F the self-dual 2-form $\mathcal{F} = \frac{1}{\sqrt{2}}(F - i * F)$. For short, we here refer to a self-dual 2-form as a *SD bivector*. The endowed metric on the 3-dimensional complex space of the SD bivectors is $\mathcal{G} = \frac{1}{2}(G - i \eta)$, η being the metric volume element of the space-time.

If F is a 2-form and P and Q are double-2-forms, $\text{Tr } P$, $P(F)$ and $P \circ Q$ denote, respectively, the scalar, the 2-form and the double-2-form given by:

$$\text{Tr } P \equiv \frac{1}{2} P_{\alpha\beta}{}^{\alpha\beta}, \quad P(F)_{\alpha\beta} \equiv \frac{1}{2} P_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}, \quad (P \circ Q)_{\alpha\beta\rho\sigma} \equiv \frac{1}{2} P_{\alpha\beta}{}^{\mu\nu} Q_{\mu\nu\rho\sigma}.$$

Moreover, we write $P^2 = P \circ P$.

Every double-2-form, and in particular the Weyl tensor W , can be considered as an endomorphism on the space of the 2-forms. The restriction of the Weyl tensor on the SD bivectors space is the *self-dual Weyl tensor* and it is given by:

$$\mathcal{W} \equiv \mathcal{G} \circ W \circ \mathcal{G} = \frac{1}{2}(W - i * W).$$

2.1 Space-time 2+2 almost-product structures

On the space-time, a 2+2 almost-product structure is defined by a time-like field of planes V and its space-like orthogonal complement H . Let v and $h = g - v$ be the respective projectors and let $\Pi = v - h$ be the *structure tensor*. A 2+2 space-time structure is also determined by the *canonical* unitary 2-form U , volume element of the time-like plane V . Then, the respective projectors are $v = U^2$ and $h = -(*U)^2$, where $U^2 = U \cdot U$.

In working with 2+2 structures it is useful to introduce the canonical SD bivector $\mathcal{U} \equiv \frac{1}{\sqrt{2}}(U - i * U)$ associated with U , that satisfies $2\mathcal{U}^2 = g$ and, consequently, $(\mathcal{U}, \mathcal{U}) = -1$. In terms of \mathcal{U} , the structure tensor is $\Pi = 2\mathcal{U} \cdot \bar{\mathcal{U}}$, where $\bar{}$ denotes complex conjugate.

The 2+2 almost-product structures can be classified by taking into account the invariant decomposition of the covariant derivative of the structure tensor Π or, equivalently, according to the foliation, minimal or umbilical character of each plane.^{27,28,29} We will say that a structure is integrable when both, V and H are a foliation. We will say that the structure is minimal (umbilical) if both, V and H are minimal (umbilical).

All these geometric properties of a 2+2 structure admit a kinematical interpretation,^{27,28} and they can be stated in terms of some first order differential concomitants of U .^{28,29} Now we summarize some of these results which we will use in the following sections. If $i(\xi)$ denotes the interior product with a vector field ξ , and δ the exterior codifferential, $\delta = *d*$, we have the following lemma:²⁸

Lemma 1 *Let us consider the 2+2 structure defined by the canonical 2-form U . The three following conditions are equivalent:*

- (i) *The structure is umbilical*
- (ii) *The canonical SD bivector $\mathcal{U} = \frac{1}{\sqrt{2}}(U - i * U)$ satisfies:*

$$\Sigma[U] \equiv \nabla \mathcal{U} - i(\xi)\mathcal{U} \otimes \mathcal{U} - i(\xi)\mathcal{G} = 0, \quad \xi \equiv \delta \mathcal{U}. \quad (1)$$

- (iii) *The principal directions of U determine shear-free geodesic null congruences.*

The umbilical condition (1) can be written, equivalently, in terms of the structure tensor $\Pi = 2\mathcal{U} \cdot \bar{\mathcal{U}}$:³³

$$\sigma[\Pi] \equiv \mathbf{s}\{2\nabla\Pi + \Pi(\nabla \cdot \Pi) \otimes \Pi - (\nabla \cdot \Pi) \otimes g\} = 0, \quad (2)$$

where $\mathbf{s}\{t\}$ stands for the total symmetrization of a tensor t , and $(\nabla \cdot \Pi)_\alpha = \nabla_\lambda \Pi_\alpha^\lambda$.

On the other hand, a 2+2 structure is minimal (respectively, integrable) if, and only if, the *expansion vector* Φ (respectively, the *rotation vector* Ψ) vanishes.²⁸ These concomitants of U are given by:

$$\begin{aligned} \Phi &\equiv \Phi[U] \equiv i(\delta U)U - i(\delta * U) * U, \\ \Psi &\equiv \Psi[U] \equiv -i(\delta U) * U - i(\delta * U)U. \end{aligned} \quad (3)$$

In terms of the structure tensor Π , the expansion and rotation vectors are, respectively:³³

$$\begin{aligned} \Phi &= \Phi[\Pi] \equiv -\frac{1}{2}\Pi(\nabla \cdot \Pi), \\ \Psi &= \Psi[\Pi] \equiv \frac{3}{2} * (\nabla \Pi \cdot \Pi), \end{aligned} \quad (4)$$

where, for a 2-tensor A and a vector x , $A(x)_\mu = A_{\mu\nu}x^\nu$, and we put $*t$ to indicate the action of the Hodge dual operator on the skew-symmetric part of a tensor t .

Finally, we show a property on the integrable directions in a 2-plane that we will use in this work. It is known^{28,29} that the 2-plane H is integrable (that is, V is a foliation) if, and only if, $i(\delta U) * U = 0$. This fact ensures that every direction in H is integrable. Now, let us suppose that H is not integrable but that H contains one integrable direction X , that is,

$$i(X)U = 0, \quad dX \wedge X = 0. \quad (5)$$

From here we obtain $(U, dX) = 0$, and then

$$(X, \delta U) = (U, dX) - \delta i(X)U = 0.$$

Then, X and $h(\delta U)$ are orthogonal directions in the 2-plane H or, equivalently, X and $i(\delta U) * U$ are collinear, that is, $X \wedge i(\delta U) * U = 0$. A similar result can be obtained by exchanging H and V . Thus, we can state:

Lemma 2 *Let U be the canonical 2-form of a 2+2 structure (V, H) . Then:*

(i) *The 2-plane V (respectively, H) is integrable if, and only if, $i(\delta * U)U = 0$ (respectively, $i(\delta U) * U = 0$).*

(ii) *If V (respectively, H) is not integrable and an integrable direction exists in V (respectively, H), then this direction is given by $i(\delta * U)U$ (respectively, $i(\delta U) * U$).*

(iii) *If X is an integrable direction in V (respectively, H), then*

$$(X, \delta * U) = 0, \quad (\text{respectively, } (X, \delta U) = 0).$$

2.2 Non null Maxwell fields

A non-null 2-form F takes the canonical expression $F = e^\phi[\cos \psi U + \sin \psi * U]$, where U is a simple and unitary 2-form that we name *geometry* of F , ϕ is the *energetic index* and ψ is the *Rainich index*. The intrinsic geometry U determines a 2+2 almost-product structure defined by the *principal planes*, the time-like one V whose volume element is U , and its space-like orthogonal complement H .

The energy (*Maxwell-Minkowski*) tensor T associated with an electromagnetic field F is minus the traceless part of its square and, in the non-null case, it depends on the intrinsic variables (U, ϕ) :

$$T \equiv -\frac{1}{2}[F^2 + *F^2] = -\frac{1}{2}e^{2\phi}[U^2 + *U^2] = -\kappa\Pi. \quad (6)$$

The symmetric tensor (6) has the principal planes of the electromagnetic field as eigen-planes and their associated eigen-values are $\pm\kappa$, with $2\kappa = \sqrt{\text{Tr } T^2} = e^{2\phi}$.

In terms of the *intrinsic elements* (U, ϕ, ψ) of a non-null Maxwell field, the source-free Maxwell equations, $\delta F = 0$, $\delta * F = 0$, take the expression:^{30,31}

$$d\phi = \Phi[U], \quad d\psi = \Psi[U]. \quad (7)$$

When F is solution of the source-free Maxwell equations, one says that U defines a *Maxwellian structure*. Besides, when the Maxwell-Minkowski energy tensor T associated with a non-null 2-form is divergence-free, the underlying 2+2 structure is said to be *pre-Maxwellian*.³² The conservation of T is equivalent to the first of the Maxwell-Rainich equations (7).³³ Then, from these equations we obtain the following result:^{30,33}

Lemma 3 (i) *A 2+2 structure is Maxwellian if, and only if, the expansion and the rotation are closed 1-forms, namely the canonical 2-form U satisfies:*

$$d\Phi[U] = 0, \quad d\Psi[U] = 0. \quad (8)$$

(ii) *A 2+2 structure is pre-Maxwellian if, and only if, the canonical 2-form U satisfies the first equation in (8).*

We can collect the expansion vector Φ and the rotation vector Ψ in a complex vector χ with a simple expression in terms of the canonical SD bivector \mathcal{U} . Indeed, we have:

$$\Phi + i\Psi = 2\chi, \quad \chi \equiv \chi[\mathcal{U}] = i(\xi)\mathcal{U}, \quad \xi \equiv \delta\mathcal{U}. \quad (9)$$

Then, conditions (8) that characterize a Maxwellian structure can be written as $d\chi = 0$.

3 Metrics admitting a non null conformal Killing-Yano 2-form A

The \mathcal{D} -metrics are Petrov-Bel type D solutions whose principal null directions define shear-free geodesic congruences. Moreover, the source of the gravitational field is a non null Maxwell field whose principal directions are those of the Weyl tensor (aligned Einstein-Maxwell solution). Thus, using the terminology introduced in Sec. 2, the Weyl principal 2+2 structure is umbilical and Maxwellian. These geometric restrictions also hold for the vacuum limit as well as for type D metrics with vanishing Cotton tensor.²⁸ Moreover, elsewhere³³ we have shown that these properties characterize the non conformally flat space-times admitting a conformal Killing-Yano 2-form. In this section we summarize what this means and we make the conditions it imposes on the curvature tensor explicit.

A Conformal Killing-Yano (CKY) 2-form is a solution A to the conformal invariant extension to the Killing-Yano equation. The conformal Killing-Yano equation takes the form:³⁴

$$\nabla_{(\alpha} A_{\beta)\mu} = g_{\alpha\beta} a_{\mu} - a_{(\alpha} g_{\beta)\mu}, \quad (10)$$

where the 1-form a is given by the codifferential of A : $3a = -\delta A$.

If A is a CKY 2-form, the scalar $\nu = A(n, p)$ is constant along an affinely parameterized null geodesic with tangent vector n , where p is a vector orthogonal to the geodesic and satisfying $n \wedge \nabla_n p = 0$. In particular, we can take $p = A(n)$, which satisfies these restrictions as a consequence of the CKY equation. Then, the scalar $A^2(n, n)$ is a quadratic first integral of the null geodesic equation, so that, the traceless part of A^2 is a second rank conformal Killing tensor, that is, a 2+2 symmetric tensor P solution to the conformal Killing equation

$$\nabla_{(\alpha} P_{\beta\mu)} = g_{(\alpha\beta} b_{\mu)}, \quad (11)$$

where the 1-form b is defined by: $3b = \nabla \cdot P$.

Elsewhere²⁹ we have shown (see also a related previous result by Hauser and Malhiot)³⁵ that a metric g which admits a 2+2 conformal Killing tensor P is conformal to a metric admitting a totally geodesic structure. This means that an umbilical and pre-Maxwellian structure exists in this space-time.^{33,29} Moreover, the full Maxwellian character of this structure is equivalent to the conformal Killing tensor being the traceless square of a CKY 2-form.³³

On the other hand, we have studied³⁶ the restrictions that the existence of a Maxwellian and umbilical (two shear-free geodesic null congruences) structure imposes on the curvature tensor. In what follows, we will go on to use some of these results which we now summarize in three lemmas:^{33,29,36}

Lemma 4 *A non conformally flat space-time admits a non null conformal Killing-Yano 2-form A if, and only if, the Weyl tensor is Petrov-Bel type D and the Weyl principal structure is umbilical and Maxwellian, that is to say, the Weyl principal bivector U satisfies:*

$$\Sigma[U] = 0; \quad d\Phi[U] = 0, \quad d\Psi[U] = 0, \quad (12)$$

where $\Sigma[U]$, $\Phi[U]$ and $\Psi[U]$ are given in (1) and (3).

Lemma 5 *If a non conformally flat space-time admits a non null conformal Killing-Yano 2-form A , and U is the principal bivector of the (type D) Weyl tensor, then two functions (ϕ, ψ) exist such that $d\phi = \Phi[U]$, $d\psi = \Psi[U]$. Moreover:*

(i) *The 2-form $F = e^\phi[\cos \psi U + \sin \psi * U]$ is a (test) Maxwell field aligned with the Weyl tensor and whose principal directions define shear-free geodesic congruences.*

(ii) *In terms of the electromagnetic variables (U, ϕ, ψ) , the intrinsic variables of the CKY tensor are $(U, -\phi/2, -\psi/2)$, that is, it takes the expression $A = e^{-(\phi/2)}[\cos(\psi/2)U - \sin(\psi/2)*U]$*

Lemma 6 *In a non conformally flat space-time which admits a non null conformal Killing-Yano tensor A the Ricci tensor R satisfies $\mathcal{S} = [R, \mathcal{U}]$, \mathcal{U} being the principal bivector of the (type D) Weyl tensor and where:*

$$\mathcal{S} = \mathcal{S}[\mathcal{U}] \equiv \mathcal{L}_\xi g - \xi \otimes \chi - \chi \otimes \xi, \quad \xi = \delta \mathcal{U}, \quad \chi = i(\xi)\mathcal{U}. \quad (13)$$

This lemma shows a close relationship between the Ricci tensor R and the Weyl principal bivector \mathcal{U} . Nevertheless, this limitation of the Ricci tensor does not necessarily restrict its algebraic type. Thus, the presence of a non null CKY tensor restricts the Weyl tensor to be of Petrov-Bel type D (or O), but the Ricci tensor may be algebraically general.

4 Metrics invariant under the complex vector $\delta\mathcal{A}$

The CKY equation (10) is invariant under the Hodge duality, so that, if A is a CKY tensor, $*A$ is a CKY tensor too. Consequently, the SD bivector $\mathcal{A} = \frac{1}{2}(A - i * A)$ satisfies the CKY equation. In terms of the electromagnetic variables (U, ϕ, ψ) of the Maxwell field given in lemma 5, the SD bivector associated with the CKY 2-form A takes the expression:

$$\mathcal{A} = e^{-\frac{1}{2}(\phi+i\psi)}\mathcal{U}, \quad \mathcal{U} \equiv \frac{1}{\sqrt{2}}(U - i * U). \quad (14)$$

Hougeston and Sommers¹⁰ showed that, for the \mathcal{D} -metric family of solutions, the divergence of the (self-dual) CKY bivector \mathcal{A} is a complex Killing vector field. Now we analyze the generic metrics where this property holds and we show that it imposes strong restrictions on the Ricci tensor.

Let us consider the complex vector $\mathcal{Z} \equiv \delta\mathcal{A}$. From (14), and considering the ξ and χ given in (9), a straightforward calculation leads to:

$$\mathcal{Z} \equiv \delta\mathcal{A} = \frac{3}{2}e^{-\frac{1}{2}(\phi+i\psi)}\xi = 3e^{-\frac{1}{2}(\phi+i\psi)}i(\chi)\mathcal{U}. \quad (15)$$

From this relation and expression (13) of \mathcal{S} , and taking into account that $d\phi + i d\psi = 2\chi$, we obtain that $\mathcal{L}_{\mathcal{Z}}g = 0$ if, and only if, \mathcal{S} vanishes identically. But lemma 6 implies that this condition states that the Ricci tensor commutes with \mathcal{U} : $[\mathcal{U}, R] = 0$. Finally, an algebraic calculation shows that this requisite restricts the Ricci tensor to be of algebraic type [(11)(11)], and the two eigen-planes are the principal planes of the bivector \mathcal{U} . Thus, we obtain:

Proposition 1 *The non conformally flat space-times admitting a conformal Killing-Yano bivector \mathcal{A} whose divergence either vanishes or is a (complex) Killing vector are those with the following properties:*

- (i) *The Weyl tensor is Petrov-Bel type D and the null principal directions define shear-free geodesic congruences.*
- (ii) *The Ricci tensor is of type [(11)(11)] and aligned with the Weyl tensor, that is,*

$$R = -\kappa\Pi + \Lambda g, \quad (16)$$

where $\Pi = 2\mathcal{U} \cdot \bar{\mathcal{U}}$, \mathcal{U} being the Weyl principal bivector.

- (iii) *There exists a (test) Maxwell field aligned with the Weyl and Ricci tensors, $F = e^\phi[\cos\psi U + \sin\psi *U]$, whose intrinsic variables (U, ϕ, ψ) determine the CKY bivector as expression (14).*

Note that the space-times characterized in the proposition above have a Ricci tensor that is, from an algebraic point of view, of (aligned) electromagnetic type $-\kappa\Pi$ plus a term proportional to the metric tensor, Λg . But the invariant scalar $\text{Tr } R = 4\Lambda$ is not, necessarily, constant and then the energy momentum tensor of the Maxwell field F is not the source of the Einstein equations, $2\kappa \neq e^{2\phi}$. Thus, the family of the \mathcal{D} -metrics is a strict subset of this family of space-times.

5 Space-times with the CKY \mathcal{A} invariant under the Killing vector $\delta\mathcal{A}$

If an Einstein-Maxwell solution admits a Killing vector Z then, as a consequence of the field equations, the electromagnetic energy tensor T is also invariant under Z . Neverthe-

less, the electromagnetic field F does not necessarily inherit the symmetry. More precisely, F satisfies:^{37,13,38}

$$\mathcal{L}_Z F = k * F, \quad (17)$$

where k is a constant when F is a non null field.^{37,13} When F is a null field, k is either a constant (if F is non-integrable) or a function that determines the front waves (if F is integrable).³⁸

Michalski and Wainwright¹³ showed that if the Einstein-Maxwell space-time admits a 2-parameter orthogonally transitive Abelian group of isometries, then the electromagnetic field inherits the symmetries. It is known that such an Abelian group exists in the case of the \mathcal{D} -metrics and, consequently, the electromagnetic field which is the source of these solutions is invariant under the Killing vectors.

Here we analyze the inheriting property from another viewpoint. In order to gain a better understanding of this behavior of the \mathcal{D} -metrics, let us consider the wider family of space-times characterized in proposition 1. In this case only the canonical geometry U is, a priori, invariant under the Killing vector \mathcal{Z} because, as a consequence of the alignment, U is a Weyl concomitant (the principal bivector). Consequently, the (test) electromagnetic field F will be invariant under \mathcal{Z} if, and only if, both, the energetic and Rainich indexes (ϕ, ψ) are.

Besides, Maxwell equations (7) imply that the energetic index ϕ (respectively, the Rainich index ψ) is invariant under \mathcal{Z} if, and only if, the expansion vector Φ (respectively, rotation vector Ψ) is orthogonal to \mathcal{Z} , $(\Phi, \mathcal{Z}) = 0$ (respectively, $(\Psi, \mathcal{Z}) = 0$).

On the other hand, from the expression (15) of the Killing vector \mathcal{Z} and from the Maxwell equations (7), and taking into account that the metric concomitant $\xi = \delta\mathcal{U}$ is invariant under the Killing vector \mathcal{Z} , one obtains:

$$[\mathcal{Z}, \bar{\mathcal{Z}}] = -(\mathcal{Z}, \bar{\chi})\bar{\mathcal{Z}}. \quad (18)$$

If Z_1 and Z_2 denote the real and imaginary parts of \mathcal{Z} , $\mathcal{Z} = Z_1 + iZ_2$, we have $[\mathcal{Z}, \bar{\mathcal{Z}}] = -2i[Z_1, Z_2]$. Consequently, from (18), \mathcal{Z} determines a commutative 2-dimensional Killing algebra (or, a unique Killing direction) if, and only if, $(\mathcal{Z}, \bar{\chi}) = 0$. As a consequence of expression (15) this orthogonality condition states, equivalently, that $\mathcal{U}(\chi, \bar{\chi}) = 0$. Taking the real and imaginary parts of this condition we get $U(\Phi, \Psi) = *U(\Phi, \Psi) = 0$. Moreover, it is evident that $\mathcal{U}(\chi, \Phi) = 0$ iff $\mathcal{U}(\Phi, \Psi) = 0$ iff $\mathcal{U}(\chi, \Psi) = 0$, that is, the expansion vector Φ is orthogonal to \mathcal{Z} , $(\Phi, \mathcal{Z}) = 0$, iff the rotation vector Ψ is, $(\Psi, \mathcal{Z}) = 0$.

Finally both, the (test) Maxwell field F and the CKY tensor A , have the same intrinsic elements (U, ϕ, ψ) and, consequently, F is invariant under the Killing vector \mathcal{Z} iff A is invariant. All these considerations allow us to state:

Proposition 2 *For the non conformally flat space-times admitting a conformal Killing-Yano bivector \mathcal{A} whose divergence $\mathcal{Z} \equiv \delta\mathcal{A} \neq 0$ is a (complex) Killing vector, the following conditions are equivalent:*

- (i) *The CKY bivector $\mathcal{A} = e^{-\frac{1}{2}(\phi+i\psi)}\mathcal{U}$ is invariant under the Killing vector \mathcal{Z} .*
- (ii) *The (test) Maxwell field $F = e^\phi[\cos\psi U + \sin\psi * U]$ is invariant under the Killing vector \mathcal{Z} .*
- (iii) *The energetic index ϕ is invariant under the Killing vector \mathcal{Z} .*
- (iv) *The Rainich index ψ is invariant under the Killing vector \mathcal{Z} .*
- (v) *The expansion vector Φ is orthogonal to the Killing vector \mathcal{Z} , $(\Phi, \mathcal{Z}) = 0$.*
- (vi) *The rotation vector Ψ is orthogonal to the Killing vector \mathcal{Z} , $(\Psi, \mathcal{Z}) = 0$.*

- (vii) $[Z_1, Z_2] = 0$, where $\mathcal{Z} = Z_1 + iZ_2$, that is, \mathcal{Z} determines either a commutative 2-dimensional Killing algebra or a unique Killing direction.
- (viii) $U(\Phi, \Psi) = *U(\Phi, \Psi) = 0$, that is, the projection of the expansion and rotation vectors on the principal planes are collinear.

Let us remark the different sort of (equivalent) points in proposition 2. The first one exclusively involves the CKY bivector \mathcal{A} : it is invariant under its divergence $\delta\mathcal{A}$.

The second one is related to the inheriting property of the (aligned) electromagnetic field and, in agreement with the Michalski and Wainwright result,¹³ it is a consequence of the commutative character of the Killing algebra (point (vii)). Nevertheless, in our reasoning here, F is a *test* electromagnetic field and not, necessarily, the source of the Einstein-Maxwell equations (the energetic index ϕ is not, a priori, a metric concomitant). Moreover, the inheriting property holds even when the Killing vector determines a unique Killing direction, that is, the existence of an Abelian group is not generically imposed although, actually, a 2-dimensional commutative algebra also exists in this degenerate case (see section 7).

Conditions (iii) and (iv) show that the invariance of a sole electromagnetic invariant scalar implies the invariance of the other one, and then of the full electromagnetic field. This fact clarifies the inheriting property for the \mathcal{D} -metrics, where the energetic index ϕ is, a priori, a metric concomitant and, consequently, it is invariant under the Killing vectors.

Maxwell equations allow us to express the above invariant properties (iii) and (iv) (differential conditions) as algebraic restrictions on the expansion and rotation vectors (conditions (v) and (vi)). Finally, condition (vii) exclusively involves the Weyl principal bivector U and, thus, it is an intrinsic restriction on the space-time.

6 Characterizing the \mathcal{D} -metrics

In the previous sections we have studied some properties that satisfy the non null CKY 2-form that the \mathcal{D} -metrics admit. Now we analyze whether these restrictions are also sufficient conditions, that is, if they characterize this family of space-times.

Lemmas 4 and 5 (Sec. III) state that the existence of the CKY 2-form in a non conformally flat space-time restrict the Weyl tensor to be Petrov-Bel type D with the two principal null directions defining shear-free geodesic congruences.

In proposition 1 (Sec. IV) we have studied the condition on the Ricci tensor to be of electromagnetic type (16) and aligned with the Weyl tensor and we have shown that it can be stated as a condition on the CKY bivector \mathcal{A} given in (14): its divergence is a complex Killing vector.

On the other hand, for the Ricci tensor (16) the contracted Bianchi identities take the form:

$$2\kappa \Phi = d\kappa - \Pi(d\Lambda). \quad (19)$$

Consequently, from the first Maxwell-Rainich equation (7), $-\kappa\Pi$ is the energy momentum tensor of the Maxwell field $F = e^\phi[\cos\psi U + \sin\psi * U]$ ($2\kappa = e^{2\phi}$) if, and only if, Λ is a (cosmological) constant. Thus, up to this simple condition on the Ricci tensor, the \mathcal{D} -metrics may be characterized with qualities of a CKY tensor. More precisely, we have:

Theorem 1 *The \mathcal{D} -metrics are the non conformally flat space-times with constant scalar curvature, $d \operatorname{Tr} R = 0$, admitting a non null CKY bivector \mathcal{A} whose divergence, $\delta \mathcal{A}$, either vanishes or is a complex Killing vector.*

It is worth remarking that the sole condition $d \operatorname{Tr} R = 0$ implies that ϕ is a metric concomitant and, consequently, it is invariant under the Killing vector $\delta \mathcal{A}$. Then, taking proposition 2 into account, we can state:

Corollary 1 *The \mathcal{D} -metrics satisfy each of the (equivalent) conditions in proposition 2. In particular, the source electromagnetic field F is invariant under the Killing vector $\delta \mathcal{A}$.*

The characterization of the \mathcal{D} -metrics given in theorem 1 is not intrinsic because it imposes conditions on a bivector \mathcal{A} and, a priori, we do not know it in terms of the metric tensor. Elsewhere⁷ we have acquired a whole intrinsic and explicit characterization of the \mathcal{D} -metrics that involves both Ricci and Weyl concomitants, and that has the suitable property of being purely algebraic in these tensors.

A whole intrinsic and explicit characterization of a metric or a family of metrics is quite interesting from a conceptual point of view - and from a practical one - because it can be tested by direct substitution of the metric tensor in arbitrary coordinates. Thus, this approach is an alternative to the usual approach to the metric equivalence problem. This and other advantages have been pointed out elsewhere³⁹ where this kind of identification has been obtained for the Schwarzschild space-time as well as for all the other type D static vacuum solutions. A similar study has been carried out for a family of Einstein-Maxwell solutions that include the Reissner-Nordström metric.²⁸

In order to obtain intrinsic and explicit characterizations, as well as having an intrinsic labeling of the metrics, we need to express these intrinsic conditions in terms of explicit concomitants of the metric tensor. When doing this, the role played by the results on the covariant determination of the eigenvalues and eigenspaces of the Ricci tensor⁴⁰ and the principal 2-forms and principal directions of the Weyl tensor^{41,42} is essential.

Now, in this section, we present another intrinsic and explicit characterization of the \mathcal{D} -metrics, which is alternative to that quoted above.⁷ The present one is differential in the curvature tensor but it has an advantage: either it involves the Weyl principal bivector \mathcal{U} and, therefore, it can be stated by means of conditions on the Weyl tensor, or it restricts the principal structure tensor Π and, therefore, it can be stated by means of conditions on the sole Ricci tensor.

Let us consider first the characterization of the \mathcal{D} -metrics by means of the principal bivector \mathcal{U} . Conditions (12) in lemma 4 assure the existence of a CKY 2-form. On the other hand, if the tensor \mathcal{S} given in (13) vanishes, then the Ricci tensor takes the expression (16), and proposition 1 implies that the divergence of the CKY bivector is a Killing vector. Therefore, from theorem 1 we obtain:

Proposition 3 *The \mathcal{D} -metrics are the non conformally flat space-times with constant scalar curvature, $d \operatorname{Tr} R = 0$, that admit a unitary bivector \mathcal{U} satisfying:*

$$\Sigma[\mathcal{U}] = 0, \quad d\chi[\mathcal{U}] = 0, \quad \mathcal{S}[\mathcal{U}] = 0, \quad (20)$$

where $\Sigma[\mathcal{U}]$, $\chi[\mathcal{U}]$ and $\mathcal{S}[\mathcal{U}]$ are given, respectively, in (1), (9) and (13).

Moreover, the space-time is Petrov-Bel type D and \mathcal{U} is the principal bivector of the Weyl tensor.

From this proposition, and considering the results given elsewhere⁴² about the invariant characterization of the Petrov-Bel type D metrics and the covariant determination of their Weyl principal bivector, we obtain:

Theorem 2 *The \mathcal{D} -metrics are the space-times with constant scalar curvature, $d \operatorname{Tr} R = 0$, whose Weyl tensor satisfies the algebraic conditions:*

$$a \neq 0, \quad \mathcal{W}^2 - \frac{b}{a} \mathcal{W} - \frac{a}{3} \mathcal{G} = 0, \quad a \equiv \operatorname{Tr} \mathcal{W}^2, \quad b \equiv \operatorname{Tr} \mathcal{W}^3, \quad (21)$$

and the differential ones (20), where \mathcal{U} is the Weyl concomitant:

$$\mathcal{U} \equiv \frac{\mathcal{P}(\mathcal{X})}{\sqrt{-\mathcal{P}^2(\mathcal{X}, \mathcal{X})}}, \quad \mathcal{P} \equiv \mathcal{W} + \frac{b}{a} \mathcal{G}, \quad (22)$$

\mathcal{X} being an arbitrary SD bivector.

A remark on the above theorem. Conditions (21) state that the space-time is of Petrov-Bel type D. Then, a previous result³⁶ implies that, under the first equation in (20) (umbilical condition), the second one (Maxwellian condition) is a consequence of its real (imaginary) part.

Let us go on the characterization of the \mathcal{D} -metrics in terms of the sole Ricci tensor. Its traceless part must satisfy the Rainich conditions:^{30,7} the algebraic ones which guarantee the type [(11)(11)], and the differential one which states that the rotation vector determines a closed 1-form. The Maxwellian character of the structure is then a consequence of the contracted Bianchi identities and, under the umbilical condition, the space-time is, necessarily, of Petrov-Bel type D.³⁶ Moreover the Weyl principal bivector is aligned with the Ricci tensor. Thus, we obtain the following Rainich-like characterization of the \mathcal{D} -metrics:

Theorem 3 *The non vacuum \mathcal{D} -metrics are the space-times whose Ricci tensor R satisfies:*

$$d \operatorname{Tr} R = 0, \quad 4\tilde{R}^2 = \operatorname{Tr} \tilde{R}^2 g \neq 0, \quad \tilde{R} \equiv R - \frac{1}{4} \operatorname{Tr} R g, \quad (23)$$

$$\sigma[\Pi] = 0, \quad d\Psi[\Pi] = 0, \quad \Pi \equiv \frac{1}{\sqrt{\operatorname{Tr} \tilde{R}^2}} \tilde{R}, \quad (24)$$

where $\sigma[\Pi]$ and $\Psi[\Pi]$ are given in (2), and (4).

The original Rainich theory³⁰ characterizes the Einstein-Maxwell solutions (without cosmological constant). In this case $\tilde{R} = R$ and the first algebraic condition in (23) must be changed to $\operatorname{Tr} R = 0$. Moreover, the algebraic Rainich conditions also impose the energy conditions on the electromagnetic energy content. This property can be added to the above theorem with a simple condition: $\tilde{R}(x, x) > 0$, where x is an arbitrary time-like vector.

It is worth remarking that all the type D vacuum solutions are \mathcal{D} -metrics that are not included in theorem 3. Although theorem 2 encompasses this vacuum limit, the simplest characterization for the type D vacuum solutions is $R = 0$ and conditions (21) which impose an algebraic type D on the Weyl tensor.

7 Solutions with a Killing tensor: the Kerr-NUT metrics

Walker and Penrose⁸ showed that a Killing tensor exists in the charged Kerr black hole, and Houghton and Sommers⁹ proved that, with the exception of the generalized charged

C-metrics, the other \mathcal{D} -metrics have also this property. From now we call the Kerr-NUT metrics the \mathcal{D} -metrics where a Killing tensor exists.

Houghston and Sommers¹⁰ showed in a subsequent work that the Killing vector \mathcal{Z} degenerates (it defines a unique real Killing vector) if, and only if, the metric admits a Killing tensor.

A later paper by Collinson and Smith¹¹ generalized a result by Floyd⁴³ and Penrose,⁴⁴ and showed that, under the assumptions of the papers by Houghston and Sommers, if a Killing tensor exists, then the space-time also admits a Killing-Yano 2-form. A similar result was also obtained by Stephani.¹²

In this section we recover all these results, and prove a set of equivalent conditions that characterize the Kerr-NUT metrics. Next, we study the Petrov-Bel type D vacuum metrics that admit a Papapetrou field aligned with the Weyl principal bivector and we show that they are also the Kerr-NUT space-times.

A *Killing-Yano* 2-form is a solution $A_{\alpha\beta}$ to the equation

$$\nabla_{(\alpha} A_{\beta)\mu} = 0. \quad (25)$$

It is known⁴ (see also references therein) that the vector $v = A(t)$ is constant along an affinely parameterized geodesic with tangent vector t . Then, the scalar v^2 is a quadratic first integral of the geodesic equation that, consequently, defines a second rank *Killing tensor*, that is, a symmetric tensor $K_{\alpha\beta}$ solution to the equation

$$\nabla_{(\alpha} K_{\beta\mu)} = 0. \quad (26)$$

In fact, this Killing tensor K is not but the square of A , $K = A^2$.

Elsewhere^{33,29} we have given the necessary and sufficient conditions that a unitary 2-form U must satisfy in order to be the geometry of either a Killing-Yano tensor³³ or a Killing tensor.²⁹ A slightly modified version of these results, more suitable for use here, is the following:

Lemma 7 *Let U be the geometry of a non null 2-form A . Then:*

(i) *A is a Killing-Yano tensor if, and only if, the bivector \mathcal{U} satisfies:*

$$\Sigma = 0, \quad d\chi = 0, \quad d\Pi(\chi) = \chi \wedge \Pi(\chi), \quad \xi \wedge \bar{\xi} = 0, \quad (27)$$

where $\Sigma \equiv \Sigma[U]$ is given in (1), $\Pi = 2\mathcal{U} \cdot \bar{\mathcal{U}}$, $\chi = i(\xi)\mathcal{U}$ and $\xi = \delta\mathcal{U}$.

(ii) *$K = A^2$ is a Killing tensor if, and only if, U satisfies:*

$$\Sigma = 0, \quad d\phi = 0, \quad d\Pi(\phi) = \phi \wedge \Pi(\phi). \quad (28)$$

where $\Phi = \Phi[U]$ is given in (3).

Note that Φ and $\Pi(\Phi)$ are the real part of χ and $\Pi(\chi)$, respectively. Then, it is easy to show that (27) implies (28), in accord with the fact that if A is a Killing-Yano 2-form then A^2 is a Killing tensor.

7.1 Characterizing the Kerr-NUT space-times

Let us consider again the family of metrics studied in section 4: those admitting a non null CKY 2-form A and invariant under the complex vector $\mathcal{Z} = \delta\mathcal{A}$.

Let us note that $\mathcal{Z} \wedge \bar{\mathcal{Z}} = -2iZ_1 \wedge Z_2$ vanishes if, and only if, \mathcal{Z} determines a unique Killing direction and, as a consequence of (15), this fact equivalently states that $\xi \wedge \bar{\xi} = 0$.

From lemma 7, this condition holds when the CKY tensor A is a full Killing-Yano tensor. We will show now that it is also a sufficient condition.

Indeed, the two first conditions in (27) hold because A is a CKY tensor. If the last one also holds, $\xi \wedge \bar{\xi} = 0$, then A will be a full Killing-Yano tensor if its geometry U satisfies the third condition in (27).

In order to prove it, we start from $\xi \wedge \bar{\xi} = 0$, which means that the Killing fields \mathcal{Z} and $\bar{\mathcal{Z}}$ are collinear and so they differ in a constant c , $\mathcal{Z} = c\bar{\mathcal{Z}}$. Then, if we make the product by \mathcal{U} and take into account that, from (15), $\mathcal{Z} = \Omega\xi$, we obtain:

$$\Omega \Pi(\chi) = c \bar{\Omega} \bar{\chi}, \quad \Omega \equiv \frac{3}{2} e^{-\frac{1}{2}(\phi + i\psi)}. \quad (29)$$

Note that $d\Omega = -\Omega\chi$. Then, the exterior derivative of the right side in (29) vanishes, and the left side becomes

$$d\Omega \wedge \Pi(\chi) + \Omega d\Pi(\chi) = 0, \quad (30)$$

and consequently, the third equation in (27) holds. Thus, we have:

Proposition 4 *In the family of non conformally flat metrics admitting a CKY 2-form A and that are invariant under the complex vector $\mathcal{Z} = \delta\mathcal{A}$, the following three conditions are equivalent:*

- (i) A is a full Killing-Yano tensor.
- (ii) \mathcal{Z} either vanishes or defines a unique Killing direction.
- (iii) The canonical bivector \mathcal{U} satisfies $\xi \wedge \bar{\xi} = 0$, $\xi \equiv \delta\mathcal{U}$.

In particular, these three conditions are equivalent for the \mathcal{D} -metrics and characterize the Kerr-NUT space-times.

If A is a Killing-Yano 2-form, then A^2 is a Killing tensor. Now we show that, in the family of metrics we are considering and, in particular, for a \mathcal{D} -metric, the converse is also true. Thus, let us suppose that the square $K = A^2$ of the CKY tensor is a Killing tensor and that \mathcal{Z} is a complex Killing vector. Then, from the definition of the expansion vector (3) and the second and third conditions in (28) we obtain:

$$di(\delta U)U = di(\delta * U)*U = i(\delta U)U \wedge i(\delta * U)*U. \quad (31)$$

Thus $i(\delta U)U$ (respectively, $i(\delta * U)*U$) is an integrable direction in the 2-plane V (respectively, H). Thus, lemma 2 implies:

$$U(\delta U, \delta * U) = 0, \quad *U(\delta U, \delta * U) = 0, \quad (32)$$

and the definitions (3) imply that Φ and Ψ are collinear on the principal planes. Then, from proposition 2 we obtain:

Lemma 8 *In the family of non conformally flat metrics admitting a CKY 2-form A and that are invariant under the complex vector $\mathcal{Z} = \delta\mathcal{A}$, if $K = A^2$ is a Killing tensor, then $(\Phi, \xi) = (\Psi, \xi) = 0$.*

In order to show that A is a Killing-Yano 2-form, we only need to prove that $\xi \wedge \bar{\xi} = 0$ as a consequence of proposition 4. We have:

$$d\Pi(\xi) = di(\xi)\bar{\mathcal{U}} = \mathcal{L}_\xi \bar{\mathcal{U}} - i(\xi)d\bar{\mathcal{U}} = \chi \wedge \Pi(\chi) - i(\xi)d\bar{\mathcal{U}}, \quad (33)$$

where we have taking into account that $\bar{\mathcal{U}}$ is invariant under the Killing vector $\mathcal{Z} = \Omega\xi$ and that $d\Omega = -\Omega\chi$. Now, taking into account the umbilical equation (1) we can compute

$$i(\xi)d\bar{\mathcal{U}} = (\xi, \bar{\chi})\bar{\mathcal{U}} - \bar{\chi} \wedge \Pi(\chi) + \frac{3}{2} i * (\xi \wedge \bar{\xi}). \quad (34)$$

Then, substituting this expression in (33) and taking its real part, we obtain:

$$d\Pi(\phi) = \phi \wedge \Pi(\phi) - \frac{3}{2}i * (\xi \wedge \bar{\xi}). \quad (35)$$

Finally, from this expression and the third condition in the characterization (28) of a Killing tensor, we obtain $\xi \wedge \bar{\xi} = 0$. In real formalism this expression states that $\delta U \wedge \delta * U = 0$. Thus:

Proposition 5 *Let us consider a non conformally flat metric admitting a CKY 2-form A such that $\mathcal{Z} = \delta A$ is a complex Killing vector. Then $K = A^2$ is a Killing tensor, if and only if, the canonical bivector U satisfies $\delta U \wedge \delta * U = 0$.*

From this result and proposition 4 we obtain the following characterizations of the Kerr-NUT space-times:

Theorem 4 *The Kerr-NUT space-times are the \mathcal{D} -metrics that satisfy one of the following equivalent conditions:*

- (i) *The CKY 2-form A is a full Killing-Yano tensor.*
- (ii) *$K = A^2$ is a Killing tensor.*
- (iii) *$\mathcal{Z} = \delta A$ either vanishes or defines a unique Killing direction.*
- (iv) *The canonical bivector U satisfies $\delta U \wedge \delta * U = 0$*

Note that the last condition in the above theorem involves the sole Weyl principal bivector U . Consequently, if we add this condition to theorem 2, we obtain an intrinsic and explicit characterization of the Kerr-NUT metrics. On the other hand, this condition may be expressed in terms of the structure tensor Π as $\Phi \wedge \Pi(\Phi) = \Psi \wedge \Pi(\Psi)$, where $\Phi \equiv \Phi[\Pi]$ and $\Psi \equiv \Psi[\Pi]$ are given in (4). Then, if we add this condition to theorem 3, we obtain an intrinsic and explicit characterization of the Kerr-NUT metrics in terms of the sole Ricci tensor.

7.2 Type D vacuum solutions with aligned Papapetrou field

If ξ is a Killing vector, the Papapetrou field $\nabla \xi$ is closed and, in the vacuum case, it is a solution of the source-free Maxwell equations.¹⁴ Metrics admitting an isometry have been studied by considering the algebraic properties of the associated Killing 2-form.^{15,16,17,18,19,20,21,22,23}

Elsewhere^{25,26} we have studied all the Petrov-Bel type I vacuum space-times admitting a Killing 2-form aligned with a principal bivector of the Weyl tensor. In the present work we fulfill a similar study for Petrov-Bel type D metrics. It is known that in the Kerr geometry this property holds,^{15,24} and we show here that the Kerr-NUT solutions are the type D vacuum metrics with a time-like Killing 2-form aligned with the Weyl geometry.

A non null 2-form F has a geometry U if, and only if, its self-dual part \mathcal{F} has the direction of U . This fact may just be stated as $[F, U] = 0$.

On the other hand, if Z is a vector field and U is a SD bivector, we have

$$\mathcal{L}_Z U = \nabla_Z U + \frac{1}{2}[dZ, U] + \frac{1}{2}\{\mathcal{L}_Z g, U\}. \quad (36)$$

From here we obtain the following result:

Lemma 9 *A Killing field Z has an associated Papapetrou field aligned with the invariant 2-form U if, and only if $\nabla_Z U = 0$.*

Let \mathcal{U} be the (invariant) principal bivector of a \mathcal{D} -metric. Then, as a consequence of the umbilical equation (1), the complex Killing vector $\mathcal{Z} = \Omega\xi$ satisfies:

$$\begin{aligned}\nabla_{\mathcal{Z}}\mathcal{U} &= \Omega \nabla_{\xi}\mathcal{U} = 0, \\ \nabla_{\bar{\mathcal{Z}}}\mathcal{U} &= (\chi, \bar{\mathcal{Z}})\mathcal{U} + \Omega i(\xi)(\bar{\xi})\mathcal{G} = \frac{1}{2}\bar{\Omega}[\xi \wedge \bar{\xi} - i*(\xi \wedge \bar{\xi})],\end{aligned}\tag{37}$$

where in the last relation we have taking into account that $(\bar{\chi}, \mathcal{Z})$ vanishes as a consequence of corollary 1 and proposition 2.

Let us consider a real Killing field Z determined by the complex one \mathcal{Z} , $Z = \mu\mathcal{Z} + \bar{\mu}\bar{\mathcal{Z}}$. From (37), $\nabla_Z\mathcal{U} = 0$ if, and only if, $\xi \wedge \bar{\xi} = 0$, that is, $\delta U \wedge \delta*U = 0$. Then, from lemma 9, we obtain:

Proposition 6 *In the set of the \mathcal{D} -metrics, the necessary and sufficient condition for a Killing field generated by the complex one \mathcal{Z} to have a Papapetrou field aligned with the principal bivector \mathcal{U} is that $\delta U \wedge \delta*U = 0$.*

This proposition and theorem 4 imply that in the Kerr-NUT space-times the complex vector $\mathcal{Z} = \delta\mathcal{A}$ determines a unique Killing direction and the associated Killing 2-form is aligned with the principal bivector. We will see below that in this case at least one other Killing vector exists and, even the dimension of the isometry group is bigger than two for the degenerate family of Kerr-NUT solutions. Nevertheless, for the generalized C-metrics, only the Killing vectors determined by \mathcal{Z} exist. Thus, we have the following result:

Theorem 5 *The Kerr-NUT solutions are the \mathcal{D} -metrics admitting a Papapetrou field aligned with the Weyl principal bivector \mathcal{U}*

We finish by recovering a known result: the \mathcal{D} -metrics admit, at least, a 2-dimensional commutative group of isometries. For the generalized C-metrics the complex vector \mathcal{Z} generates a 2-dimensional commutative algebra. For the Kerr-NUT metrics it only generates a Killing direction Z . But in this case, if K is the Killing tensor, $Y = K(Z)$ is another Killing vector that commutes with Z .

Indeed, from the Killing tensor equation (26), for an arbitrary vector field Z we have:

$$\mathcal{L}_Z K - 2[\nabla Z, K] + \mathcal{L}_{K(Z)}g = 0.$$

The first summand $\mathcal{L}_Z K$ vanishes because $K = A^2$ is invariant under Z . The second summand $[\nabla Z, K]$ also vanishes because, as a consequence of the proposition 6, the Killing 2-form ∇Z is aligned with \mathcal{U} , that is, $[\nabla Z, \mathcal{U}] = 0$ and then, ∇Z commutes with $K = \mu\Pi + \nu g$. Thus, the metric is invariant under $Y = K(Z)$, a vector field that commutes with Z because K is invariant.

Note that Z and $Y = A^2(Z)$ could define a unique Killing direction. This means that Z is eigenvector of A^2 with constant eigenvalue. If we impose this condition and consider the expression of Z and A in terms of U , we obtain that these degenerate Kerr-NUT metrics are characterized by one of the following conditions:

$$i(\delta U)U = i(\delta*U)U = 0, \quad \text{or} \quad i(\delta U)*U = i(\delta*U)*U = 0.\tag{38}$$

Under the Kerr-NUT requisite $\delta U \wedge \delta*U = 0$, it is easy to show that (38) does not hold if

$$i(\delta U)U \wedge i(\delta U)*U \neq 0, \quad \text{or} \quad i(\delta*U)U \wedge i(\delta*U)*U \neq 0.\tag{39}$$

Elsewhere⁴⁵ we undertake the intrinsic labeling of the \mathcal{D} -metrics and analyze in detail these degenerate Kerr-NUT metrics. We show in that work⁴⁵ that they admit a 4-dimensional group of isometries with a 2-dimensional commutative subgroup, accordingly with the known literature on this subject.^{3,4} Taking into account all these considerations, we can state:

Proposition 7 *The \mathcal{D} -metrics admit, at least, a commutative 2-dimensional group of isometries. Let \mathcal{U} and \mathcal{A} be the canonical and the CKY bivectors. Then:*

(i) *If $\delta U \wedge \delta *U \neq 0$ (generalized C-metrics), the two Killing vectors are determined by $\mathcal{Z} = \delta \mathcal{A}$.*

(ii) *If $\delta U \wedge \delta *U = 0$ and (39) (Kerr-NUT regular metrics), one Killing vector Z is determined by \mathcal{Z} , and the other one is $Y = A^2(Z)$.*

(iii) *If $\delta U \wedge \delta *U = 0$ and (38) (Kerr-NUT degenerate metrics), the Killing vector Z determined by \mathcal{Z} belongs to a 4-dimensional Killing algebra.*

Acknowledgements

This work has been partially supported by the Spanish Ministerio de Educación y Ciencia, MEC-FEDER project FIS2006-06062.

References

- ¹ J. F. Plebański and M. Demiański, Ann. Phys. (NY) **98**, 98 (1976).
- ² G. J. Weir, and R.P. Kerr, Proc. Roy. Soc. A **355**, 31 (1977).
- ³ R. Debever, N. Kamran, and R. C. McLenaghan, J. Math. Phys. **25**, 1955 (1984).
- ⁴ E. Stephani, H. Kramer, M. A. H. McCallum, C. Hoenselaers and E. Hertl, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 2003).
- ⁵ J. B. Griffiths and J. Podolský, Int. J. Mod. Phys. D **15**, 335 (2006).
- ⁶ W. Kinnersley and M. Walker, Phys. Rev. D **2**, 1359 (1970).
- ⁷ J. J. Ferrando and J. A. Sáez, Gen. Relativ. Gravit. **39**, xxx (2007).
- ⁸ M. Walker and R. Penrose, Commun. Math. Phys. **18**, 265 (1970).
- ⁹ L. P. Houghston and P. Sommers, Commun. math. Phys. **32**, 147 (1973).
- ¹⁰ L. P. Houghston and P. Sommers, Commun. math. Phys. **33**, 129 (1973).
- ¹¹ C. D. Collinson and P. N. Smith, Commun. math. Phys. **56**, 277 (1977).
- ¹² H. Stephani, Gen. Relativ. Gravit. **9**, 789 (1978).
- ¹³ H. Michalski and J. Wainwright, Gen. Relativ. Gravit. **5**, 289 (1975).
- ¹⁴ A. Papapetrou, Ann. Inst. H Poincaré A **4**, 83 (1966).
- ¹⁵ F. Fayos and C. F. Sopuerta, Class. Quantum Grav. **16**, 2965 (1999).
- ¹⁶ G. C. Debney, J. Math. Phys. **12**, 1088 (1971).
- ¹⁷ G. C. Debney, J. Math. Phys. **12**, 2372 (1971).
- ¹⁸ C. B. G. McIntosh, Gen. Rel. Grav. **7**, 199 (1976).
- ¹⁹ C. B. G. McIntosh, Gen. Rel. Grav. **7**, 215 (1976).
- ²⁰ F. Fayos and C. F. Sopuerta, Class. Quantum Grav. **18**, 353 (2001).
- ²¹ J. D. Steele, Class. Quantum Grav. **19**, 259 (2002).
- ²² G. Ludwig, Class. Quantum Grav. **19**, 3799 (2002).
- ²³ F. Fayos and C. F. Sopuerta, Class. Quantum Grav. **19**, 5489 (2002).
- ²⁴ M. Mars, Class. Quantum Grav. **17**, 3353 (2000).
- ²⁵ J. J. Ferrando and J. A. Sáez, Class. Quantum Grav. **20** 5291 (2003).
- ²⁶ J. J. Ferrando and J. A. Sáez, J. Math. Phys. **47**, 112501 (2006).
- ²⁷ B. Coll and J. J. Ferrando, “Almost-Product structures in Relativity” in Recent developments in Gravitation, *Proceeding of the Relativist Meeting-89* (World Scientific, Singapore, 1990), p. 338.

- ²⁸ J. J. Ferrando and J. A. Sáez, J. Math. Phys. **45**, 652 (2004).
- ²⁹ B. Coll, J. J. Ferrando and J. A. Sáez, J. Math. Phys. **47**, 062503 (2006).
- ³⁰ G. Y. Rainich, Trans. Am. Math. Soc. **27**, 106 (1925).
- ³¹ B. Coll, F. Fayos and J. J. Ferrando, J. Math. Phys. **28**, 1075 (1987).
- ³² R. Debever, Bull. Acad. R. Belgique **62**, 662 (1976).
- ³³ J. J. Ferrando and J. A. Sáez, Gen. Relativ. Gravit. **35**, 475 (2003).
- ³⁴ S. Tachibana, Tôhoku Math. J. **21**, 56 (1969).
- ³⁵ J. Hauser and R. J. Malhiot, J. Math. Phys. **19**, 187 (1978).
- ³⁶ J. J. Ferrando and J. A. Sáez, Gen. Relativ. Gravit. **39**, 343 (2007).
- ³⁷ J. R. Ray and E. L. Thompson, J. Math. Phys. **16**, 345 (1975).
- ³⁸ B. Coll, C. R. Acad. Sc. Paris **280**, 1773 (1975).
- ³⁹ J. J. Ferrando and J. A. Sáez, Class. Quantum Grav. **15**, 1323 (1998).
- ⁴⁰ C. Bona, B. Coll and J. A. Morales, J. Math. Phys. **33**, 670-81 (1992).
- ⁴¹ J. J. Ferrando and J. A. Sáez, Class. Quantum Grav. **14**, 129 (1997).
- ⁴² J. J. Ferrando, J. A. Morales and J. A. Sáez, Class. Quantum Grav. **18**, 4969 (2001).
- ⁴³ R. Floyd, "The Dynamics of Kerr fields". (PH.D. Thesis, London University, 1973).
- ⁴⁴ R. Penrose, Ann. N. Y. Acad. Sci. **224**, 125 (1973).
- ⁴⁵ J. J. Ferrando, and J. A. Sáez, (in preparation)